Imperial College
London

IMPERIAL COLLEGE LONDON

DEPARTMENT OF THEORETICAL PHYSICS

Traversable Wormhole Constructions

Author:
Catalina-Ana Miritescu

Supervisor:
Professor Toby Wiseman

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Abstract

This paper presents a history of wormholes from the beginning of the field to the most recent developments. The conditions necessary for a traversable wormhole's existence are introduced, with a particular focus on ways in which violations of the averaged null energy condition (ANEC) can be achieved. A traversable wormhole construction is presented in detail including geometric assembly, ANEC violation through Casimir-like negative energy due to fermionic fields, and wormhole stabilization through mouths' rotation around each other. Other recent constructions are then briefly discussed and compared to the first one.

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Section 1

Introduction

Both the scientific community and the wide public have been fascinated by wormholes for decades. This concept appears in numerous scientific papers and in countless science-fiction artistic creations: movies (“Interstellar”), books (a notable example is the novel “Contact” by Carl Sagan, whose questions were the source of the theories developed later on by Kip Thorne), TV series (“Dark”, “Stargate”), paintings and songs. There is not a single person whose imagination, creativity and curiosity are not incited by the term “wormhole”. One of the trademarks of humankind is its desire to explore its surrounding environment, so it is only natural that the possibility of faster-than-light travel offered by wormholes - at least in popular belief - is seen as an invaluable resource. Achieving this would bring about a new era of space exploration: discoveries of habitable worlds located in our own galaxy or even in other galaxies, possible contact with alien life forms and the possibility to test our theorized laws of physics in the furthest corners of the Universe.

Besides travel through space over great distances in the blink of an eye, the idea of a wormhole is connected to the possibility of travel back through time. This concept has been the basis for multiple paradoxes, perhaps the most familiar one being the so called “grandfather paradox”. It can be summarized in the following way: if a person goes back in time and kills their own grandparents before the conception of their parents, would the killer instantaneously stop existing? And if that happens, would the grandparents remain alive, since there is no one to travel back in time and end their lives now?

From a scientific point of view, the possible existence of wormholes draws attention to questions related to causality, the geometric structure and topology of spacetime, quantum gravity, and energy constraints. All these topics will be discussed in the chapters below.

This paper aims to provide some clarity about the concept of wormholes. We start by defining wormholes and various associated terminology in the second section. The third section is dedicated to the history of wormholes as seen through scientific publications and the many ways in which they have been theorized in the past. We make a clear distinction between non-traversable and traversable wormholes. The fourth section summarizes the necessary conditions to create and stabilize a traversable wormhole, and the difficulties encountered in fulfilling them. In the fifth section, we address the most recent papers which provide a recipe to produce stable, traversable wormholes. The last section is reserved for open questions and closing remarks.

Section 2

Terminology

A wormhole can be roughly defined as a tunnel connecting two different regions of spacetime. More specifically, Visser defines a wormhole in his 1995 reference book "Lorentzian Wormholes from Einstein to Hawking" to be "any compact region of spacetime with a topologically simple boundary, but a topologically nontrivial interior" [1]. Given the location of the two regions, we can characterize wormholes as being either intra-universe wormholes (they connect two regions which belong to the same universe) or inter-universe wormholes (they connect two regions which belong to two different universes). Given the manifold in which they reside, we can further split them into Lorentzian (pseudo-Riemannian) or Euclidean (true Riemannian) wormholes. Experimentally, real physics seems to take place in Lorentzian signature. Furthermore, based on the amount of time the wormhole remains opened, we can distinguish between permanent (more realistically quasipermanent), long-lived wormholes or transient, short-lived types, which pop in and out of existence[1].

An important characteristic of a wormhole is its traversability; if a particle can enter through one side of the wormhole and it can exit through the other, the wormhole is traversable. If this cannot happen, but two particles entering from opposite sides of the wormhole meet somewhere in the tunnel, the wormhole is non-traversable. Based on the amount of time it takes to go from point A to point B through the wormhole, compared to the amount of time it takes to reach point B from point A in the regular spacetime, wormholes are characterized as either short - going from A to B through the ambient space takes longer than going through the wormhole – or long, if the inverse is true (Figure 2.1).

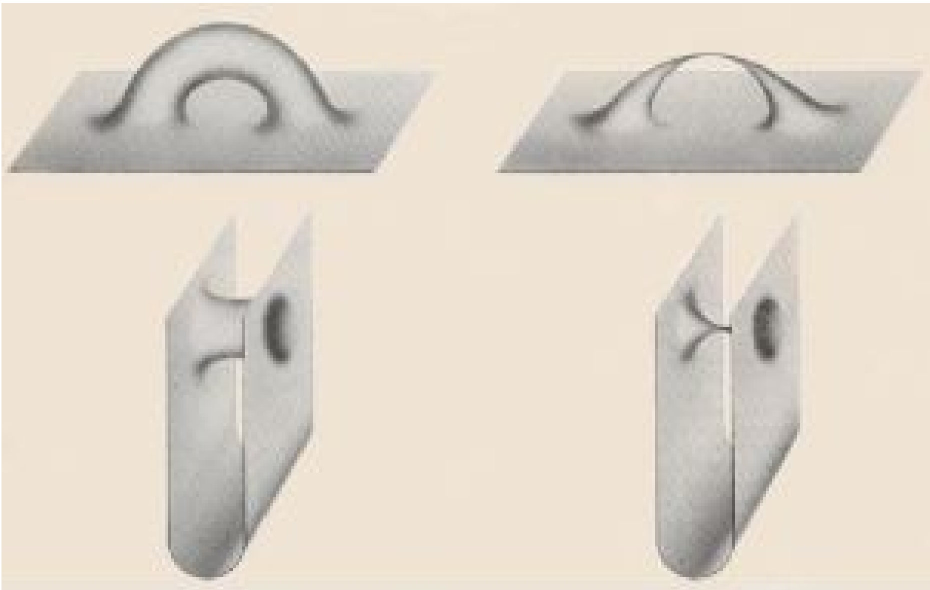

Figure 2.1: Two examples of long wormholes (top) and two examples of short wormholes (bottom) with different throat circumferences. Figure taken from reference [2].

Since this definition only makes sense for traversable wormholes, traversability is implied when talking about short or long wormholes. Inter-universe wormholes are by default short, since going from one universe to another without the aid of the wormhole is impossible – it would take infinite time. If a wormhole exists on every constant-time hypersurface in a spacetime which admits a time function, the wormhole is called “eternal”. It is important to note that an eternal wormhole might only be traversable for a certain amount of time, and a change of traversability does not require a change of topology. If an eternal wormhole is traversable for all time, then the wormhole is “eternally traversable”.

Based on the circumference of the “mouth” relative to the Planck length, wormholes can be further characterized as macroscopic or microscopic [1].

An important term which appears in all conversations about wormholes is the wormhole's throat. In a paper published in 1998 by D. Hochberg and M. Visser, the throat is defined as "a two–dimensional hypersurface of minimal area" [3], or the point where the radius is minimal, for a static wormhole. However, for a dynamical wormhole, this definition cannot be restricted to only one slice of time. Hochberg's and Visser's definition in this case is "a closed two–dimensional spatial hypersurface such that one of the two future-directed null geodesic congruences orthogonal to it is just beginning to diverge", which they call a marginally anti–trapped surface [3]. This is closely related to Raychaudhuri equation [4], which will be stated and explained in detail in section 4.2.

Section 3

History of wormholes

3.1 Non-traversable wormholes

The first theoretical example of a non-traversable wormhole comes from the Schwarzschild solution to the equations of Einstein's general theory of relativity. In 1916 Ludwig Flamm realized that, besides the already known Schwarzschild black hole solution, there exists a second, simple solution, which is now known as a white hole. The two solutions, describing two different regions of (flat) spacetime are connected (mathematically) by a spacetime conduit [5]. This idea was further explored in 1935 by Albert Einstein and Nathan Rosen in a paper whose primary focus was actually the development of "an atomistic theory of matter and electricity which, while excluding singularities of the field, makes use of no other variables than the $g_{\mu\nu}$ of the general relativity theory and the ϕ_μ of the Maxwell theory" [6], giving rise to the Einstein-Rosen bridge (Figure 3.1). In order to understand this structure mathematically and topologically, we need to first consider the metric of the solution to the Einstein's field equations and the corresponding coordinate systems.

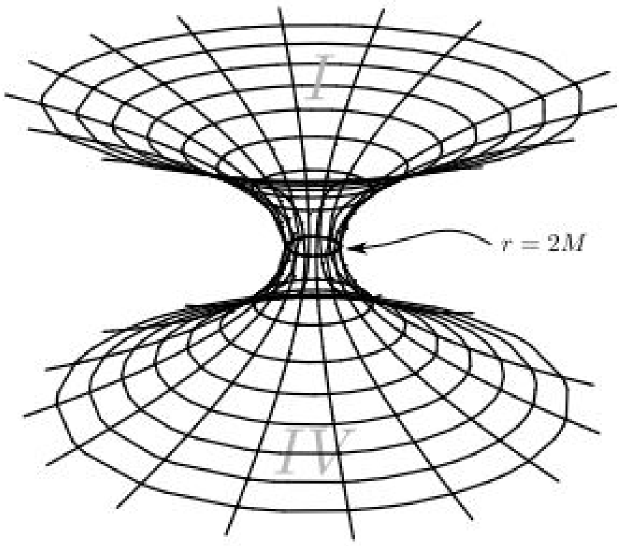

Figure 3.1: An Einstein-Rosen bridge with one dimension suppressed. Each circle in the two-dimensional space represents a two-sphere in the three-dimensional analogue. Figure taken from reference [7].

The corresponding metric in the simplest case (spherical symmetry, no electric and no magnetic charge) is the Schwarzschild solution to Einstein's field equations:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2),$$

where $r > 2M$, θ goes from 0 to π, and ϕ varies from 0 to 2π. There appears to be a singularity at $r = 2M$, but this is only a coordinate artifact arising from choosing an unfortunate coordinate system. This apparent singularity can be removed by switching to a more appropriate coordinate system, such as the Eddington–Finkelstein (EF) coordinates: ingoing EF coordinates for black holes, outgoing EF coordinates for white holes [8]. The ingoing coordinates are obtained by replacing t by $t = v - r^*$, where r^* is defined such that:

$$\frac{dr^*}{dr} = \left(1 - \frac{2GM}{r}\right)^{-1}$$

The coordinate transformation $(t, r, \theta, \phi) \rightarrow (v, r, \theta, \phi)$ allows us to rewrite the metric as:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

For the outgoing EF coordinates, the change in coordinates $(t, r, \theta, \phi) \to (u, r, \theta, \phi)$ is given by $t = u + r^*$ and the obtained metric is:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

However, both white holes and black holes can be covered by the same set of coordinates called the Kruskal–Szekeres (KS) coordinates - they cover the whole manifold without running into any singularity artifacts. The KS coordinates are usually denoted by (U, V, θ, ϕ) and are defined as:

$$U = -e^{-u/4M} \qquad V = e^{v/4M}$$

Initially the metric is defined only for $U < 0$ and $V > 0$, but we can analytically extend it to obtain what is called the maximally extended Schwarzschild solution:

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dUdV + r^2(d\theta^2 + sin^2\theta d\phi^2),$$

where $-\infty < U, V < \infty$ and $r = r(V, U)$ is given implicitly by:

$$UV = -\left(\frac{r - 2M}{2M}\right) e^{r/2M}$$

This gives rise to the Kruskal diagram (Figure 3.2). The whole manifold is split into 4 regions depending on the sign of U and V. The regions are usually numbered using Roman numerals. Region I is the original $r > 2M$ section in Schwarzschild coordinates. Regions I and II are also covered by the ingoing EF coordinates and they are relevant to black hole geometry, while regions I and III are covered by the outgoing EF coordinates and are relevant to white holes. Region IV is a new region, isometric to region I under the $(U, V) \to (-U, -V)$ transformation. The singularity at $r = 0$ corresponds to $UV = 1$, while the $r = 2M$ lines correspond to $UV = 0$ - either $U = 0$ or $V = 0$. Each point of the diagram can be viewed as representing a 2-sphere of radius r. Alternatively, the diagram can be interpreted as the causal structure of radial motion for fixed θ, ϕ polar angles.

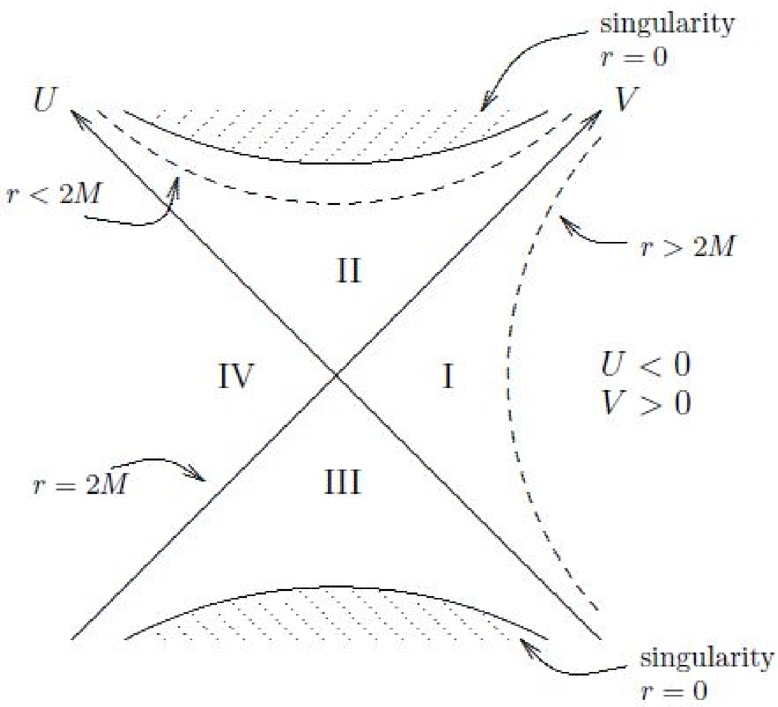

Figure 3.2: A Kruskal diagram with its 4 separate regions. Figure taken from [9].

In region I we have $\frac{U}{V} = e^{-t/2M}$, so constant Schwarzschild time slices of spacetime are represented in the Kruskal diagram through straight lines passing through the origin. These hypersurfaces have a part in region I and a part in region IV. It is easier to look at this geometry in a different coordinate system, the so called isotropic coordinates (t, ρ, θ, ϕ), where ρ is the new radial coordinate:

$$r = \left(1 + \frac{M}{2\rho}\right)^2 \rho = \rho + M + \frac{M^2}{4\rho^2}$$

The metric in this case is:

$$ds^2 = -\left(\frac{1 - \frac{M}{2\rho}}{1 + \frac{M}{2\rho}}\right)^2 dt^2 + \left(1 + \frac{M}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega^2)$$

Given r, there are two solutions for ρ. The two values of ρ are exchanged by the isometry $\rho \rightarrow \frac{M^2}{4\rho}$ which has $\rho = \frac{M}{2}$ as its fixed 'point'. This is actually a fixed 2-sphere of radius $2M$. This isometry interchanges regions I and IV and it is equivalent to the $(U, V) \rightarrow (-U, -V)$ transformation. We consider $\rho > \frac{M}{2}$ for region I, and $\frac{M}{2} > \rho > 0$ for region IV. The isotropic coordinates cover only regions I and IV since ρ is complex for $r < 2M$ (Figure 3.3).

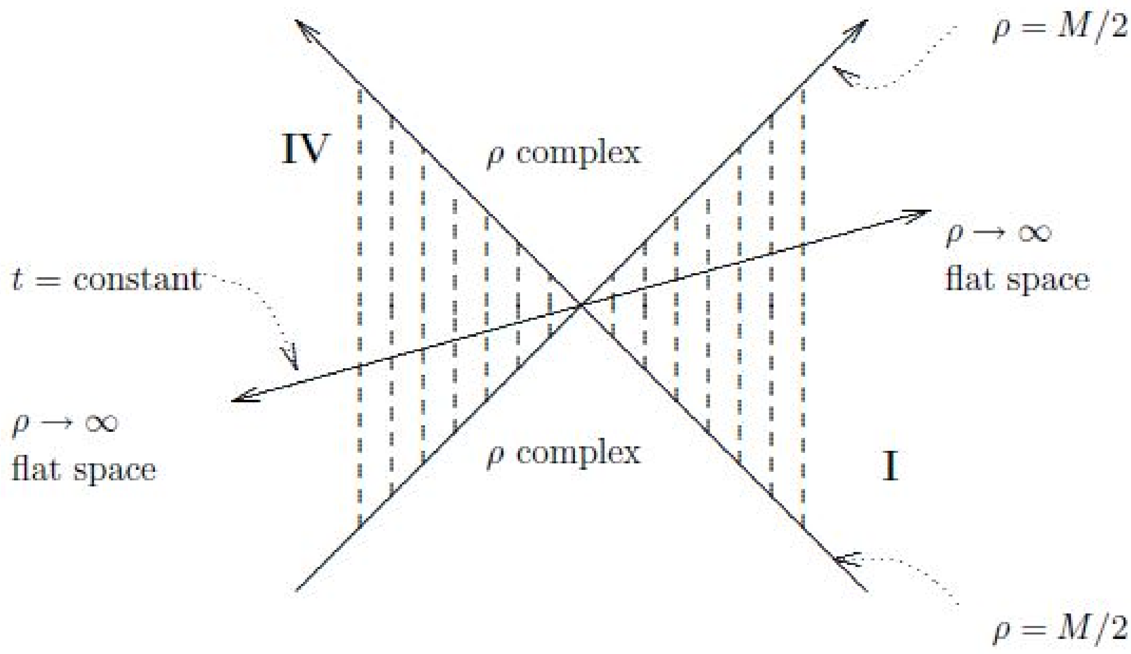

Figure 3.3: Isotropic coordinates on a Kruskal diagram. Figure taken from [9].

For the $t = constant$ geometry, as we approach $\rho = \frac{M}{2}$ from either side, the radius of the 2-sphere representing each point decreases to minimum of $r = 2M$ at $\rho = \frac{M}{2}$ - this is the minimal 2-sphere. There are two asimptotically flat regions at $\rho \rightarrow \infty$ and $\rho \rightarrow 0$, which are connected by a throat with the corresponding minimum radius equal to that of the minimal 2-sphere: $2M$. This throat is mathematically what we call the Einstein-Rosen bridge (Figure 3.1).

It is tempting to say that this is a traversable wormhole which connects regions I and IV. However, this is a snapshot at a constant time t, so it is not possible to travel through it - thus it is a non-traversable wormhole.

The field of study remained mostly dormant for twenty years after this discovery. The interest was rekindled by the physicist John Wheeler in 1955. His paper coined the term “wormholes” and it discussed them in terms of topological entities called geons (unstable but long lived solutions to the combined Einstein-Maxwell equations). It also provided the first (now familiar) diagram of a wormhole as a tunnel connecting two openings in different regions of spacetime [10] (Figure 3.4).

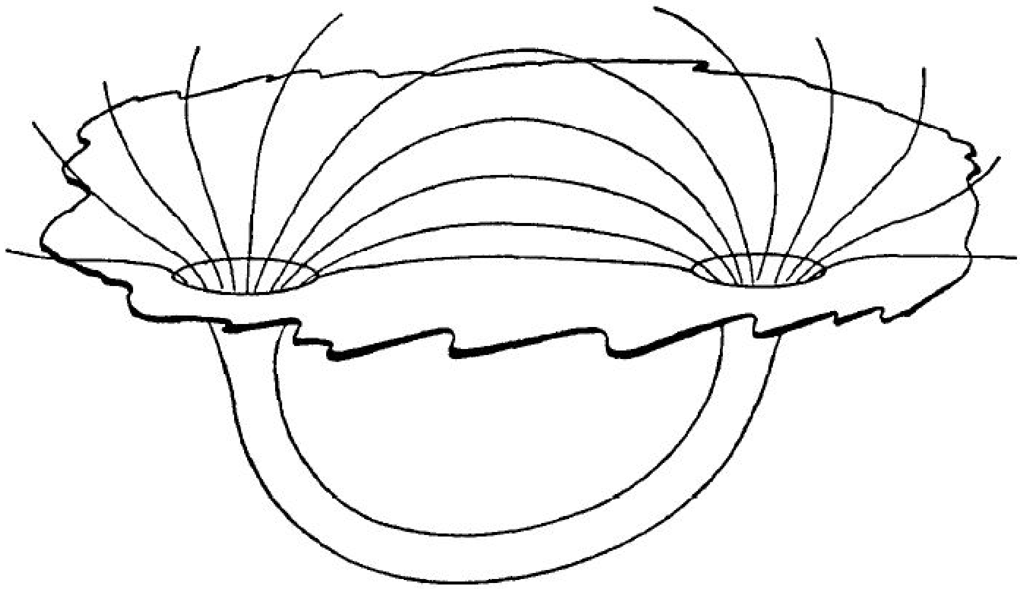

Figure 3.4: Wheeler’s schematic drawing of a wormhole - “Schematic representation of lines of force in a doubly-connected space”. Drawing taken from reference [10].

Wheeler wormholes are microscopic in nature, since their occurrence is allegedly due to vacuum fluctuations in the spacetime foam. They are typically transient, though there might exist situations when their topology is suitable to be considered quasipermanent [1].

3.2 Traversable wormholes

1973 was an important year for traversable wormholes. Independently, both Homer G. Ellis [11] and Kirill A. Bronnikov [12] published papers demonstrating the possibility of traversable wormholes in general relativity. The Ellis drainhole has been regarded to be the earliest complete model of such a wormhole. It is a static, spherically symmetric solution of the Einstein vacuum field equations, combined with a scalar field ϕ minimally coupled to the geometry of space-time with opposite coupling polarity (negative instead of positive) [11]:

$$\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{R}g_{\mu\nu} = 2\left(\phi_{,\mu}\phi_{,\nu} + \frac{1}{2}\phi^{,\sigma}\phi_{,\sigma}g_{\mu\nu}\right)$$

The metric in this case is:

$$ds^2 = -dt^2 + (d\rho - f(\rho)dt)^2 + r^2(\rho)(d\theta^2 + sin^2\theta d\phi^2)$$

$$ds^2 = -[1 - f^2(\rho)]dT^2 + \frac{1}{1 - f^2(\rho)}d\rho^2 + r^2(\rho)(d\theta^2 + sin^2\theta d\phi^2)$$

where

$$T = t + \int \frac{f(\rho)}{1 - f^2(\rho)}d\rho,$$

the functions $f(\rho)$ and $r(\rho)$ are to be determined by the field equations, and the coordinate ranges are given by:

$$-\infty < t < \infty; -\infty < \rho < \infty; 0 < \theta < \pi; -\pi < \phi < \pi.$$

The solutions depend on two parameters, usually denoted by m and n, which have to satisfy the inequalities $0 \leq m < n$, but are otherwise unconstrained. In terms of these parameters, the functions are:

$$f(\rho) = -\sqrt{1 - e^{-(2m/n)\Phi}};$$

$$r(\rho) = \sqrt{(\rho - m)^2 + a^2} e^{(m/n)\Phi},$$

in which $\Phi = \frac{n}{a}\left[\frac{\pi}{2} - tan^{-1}\left(\frac{\rho - m}{a}\right)\right]$ and $a = \sqrt{n^2 - m^2}$ [11].

The movement of particles through this wormhole is similar to the movement of liquid on turbulent draining curves (Figure 3.5). Depending on the direction of the initial velocity of each particle, we can obtain different trajectories. For example, particle 4 goes straight toward the wormhole on the left side, then right through the wormhole and exits on the left side, while particle 2 enters the wormhole on the right side and spirals around the center of the wormhole at least once before exiting the wormhole on the left side.

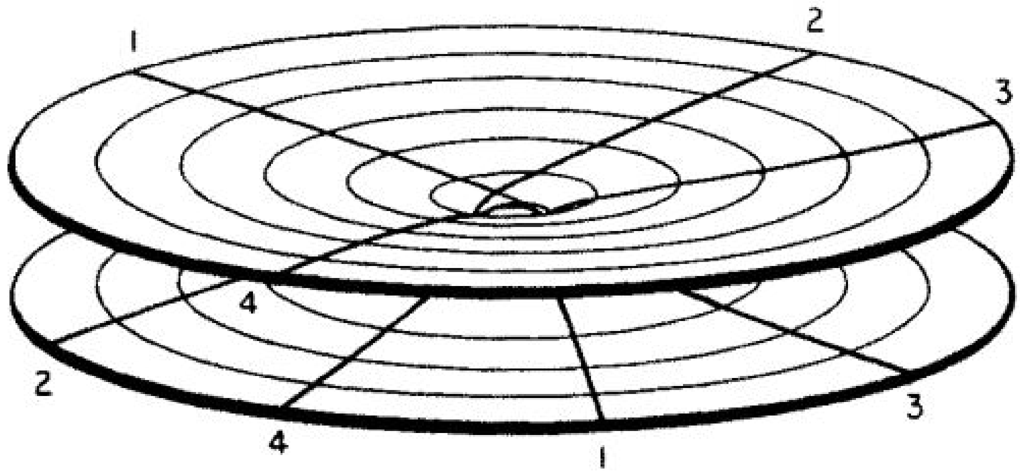

Figure 3.5: Example of an Ellis wormhole and 4 different particle trajectories going through it. Image taken from [11].

The development of this field continued with the publication of two papers authored by Kip Thorne, Michael Morris, and Uri Yertsever in 1988 [13], [14], which explained the so called Morris-Thorne (MT) wormhole. They present nine desirable properties for a long-lived traversable wormhole, some mandatory for its existence, some chosen only to simplify calculations [14]:

1. The metric should be spherically symmetric and time-independent (static). This condition was selected to keep calculations simple. The general form

of such a metric is:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Omega^2,$$

where $d\Omega^2 = sin^2\theta d\phi^2 + d\theta^2$ and $\Phi(r)$ and $\lambda(r)$ are unknown functions of r. The expression of the MT wormhole metric is:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2,$$

where $\Phi(r)$ and $b(r)$ are functions of r which will be constrained by the properties bellow. The function $b(r)$ determines the spatial shape of the wormhole, and thus it is called the "shape function", while $\Phi(r)$ determines the gravitational redshift and it is called the "redshift function".

2. The solution must obey the Einstein field equations everywhere:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

3. "The solution must have a throat that connects two asymptotically flat regions of space–time" [14].

4. No horizon can exist - the presence of one would prevent two-way travel through the wormhole. This restriction corresponds to $\Phi(r)$ finite everywhere.

5. The tidal gravitational forces experienced by a traveler must be bearably small.

6. The traveler must be able to cross through the wormhole in a finite and reasonably small proper time.

7. The matter and fields which create the wormhole's spacetime curvature must have a physically reasonable stress-energy tensor.

8. The solution should be stable under small perturbations.

9. It should be possible to assemble the wormhole. For example, the mass needed to create the wormhole should be much smaller than the total mass of the universe, and the time it takes to occur should be much smaller than the age of the universe.

This paper calls the first four properties “basic wormhole criteria” [14], while properties five and six - which change the wormhole’s parameters for human physiological comfort - are called “usability criteria” [14].

Before calculating any components of the Einstein tensor, we switch to a different coordinate system which will greatly simplify our calculations. We choose a basis of orthonormal vectors - the “proper reference frame of a set of observers who remain always at rest in the coordinate system $(r, \theta, \phi = constant)$” [14]:

$$e_t = e^{-\Phi} e_{t_0} \qquad e_r = \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} e_{r_0}$$

$$e_\theta = r^{-1} e_{\theta_0} \qquad e_\phi = (r\ \sin\theta)^{-1} e_{\phi_0},$$

where $e_{t_0}, e_{r_0}, e_{\theta_0}, e_{\phi_0}$ are the initial orthonormal basis vectors:

$$e_{t_0} = \frac{\partial}{\partial t} \qquad e_{r_0} = \frac{\partial}{\partial r}$$

$$e_{\theta_0} = \frac{\partial}{\partial \theta} \qquad e_{\phi_0} = \frac{\partial}{\partial \phi}$$

In the new basis, the metric coefficients take on their standard, special relativity form:

$$g_{\alpha\beta} = e_\alpha e_\beta = \eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All the calculations following this point will be done in the new coordinate basis.

By calculating the Riemann tensor $R^{\mu}_{\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and by solving the Einstein field equations, we obtain the following equalities:

$$G_{tt} = \frac{b_{,r}}{r^2}$$

$$G_{rr} = -\frac{b}{r^3} + 2\left(1 - \frac{b}{r}\right)\frac{\Phi_{,r}}{r}$$

$$G_{\theta\theta} = \left(1 - \frac{b}{r}\right)\left(\Phi_{,rr} - \frac{b_{,r}r - b}{2r(r-b)}\Phi_{,r} + (\Phi_{,r})^2 + \frac{\Phi_{,r}}{r} - \frac{b_{,r} - b}{2r^2(r-b)}\right) = G_{\phi\phi}$$

Furthermore, we can denote the stress-energy tensor components in the following way:

$$T_{tt} = \rho(r)$$

$$T_{rr} = -\tau(r)$$

$$T_{\theta\theta} = T_{\phi\phi} = p(r),$$

where $\rho(r)$ is the total mass-energy density, $\tau(r)$ is the radial tension per unit area, and $p(r)$ is the pressure in the lateral direction. Using the equality $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$, we obtain:

$$\rho(r) = \frac{b_{,r}}{8\pi G r^2}$$

$$\tau(r) = \frac{\frac{b}{r} - 2(r-b)\Phi_{,r}}{8\pi G r^2}$$

$$p(r) = \frac{r}{2}\left[(\rho(r) - \tau(r))\Phi_{,r} - \tau_{,r}\right] - \tau(r)$$

In this form, by choosing a suitable $b(r)$ and $\Phi(r)$ (according to the constraints above), we will be able to solve for $\rho(r)$ and $\tau(r)$. The strongest constraints occur at the throat of the wormhole. There, the tension must be larger than the total density of mass-energy $\tau_0 > \rho_0$ [14]. Materials which have this specific property are called "exotic". This relation is problematic because of its implications for measurements made by an observer falling through the wormhole with a radial velocity close to the speed of light. If such an observer moves sufficiently fast, they will see a negative matter-energy density [14]. The case where a static observer sees a negative matter-energy density $\rho_0 < 0$ comes as a natural extension of the situation presented above. In the following section, specific circumstances in which this condition is satisfied will be explored.

Section 4

Constraints

4.1 Energy conditions

There are seven energy conditions which are useful when talking about matter and energy in general relativity. They are [1]:

1. The null energy condition (NEC) - given the stress-energy tensor $T_{\mu\nu}$, for any null vector k_μ, when NEC is satisfied the following relation holds:

$$T_{\mu\nu}k^\mu k^\nu \geq 0$$

2. The weak energy condition (WEC) - for any timelike vector V^μ, the following relation is true:

$$T_{\mu\nu}V^\mu V^\nu \geq 0$$

If this is true for any timelike vector, it will imply the null energy condition. The physical importance of this condition is that it forces the local energy density as measured by any timelike observer to be positive.

$$\text{WEC} \Rightarrow \rho \geq 0$$

3. The strong energy condition (SEC) - any timelike vector V^μ must satisfy the relation:
$$\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right)V^\mu V^\nu \geq 0,$$
where T is the trace of the stress-energy tensor $T = T_{\mu\nu}g^{\mu\nu}$. The strong energy condition implies the null energy condition, but it does not, in general, imply the weak energy condition.

4. The dominant energy condition (DEC) - for any timelike vector V^μ, the following relations are true:
$$T_{\mu\nu}V^\mu V^\nu \geq 0$$
and $T_{\mu\nu}V^\nu$ is not spacelike. The dominant energy condition implies the weak energy condition, and thus it implies the null energy condition. For our purpose, the important condition is:
$$\text{DEC} \Rightarrow \rho \geq 0$$

5. The averaged null energy condition (ANEC) - this condition holds if on all null curves Γ we have:
$$\int_\Gamma T_{\mu\nu}k^\mu k^\nu d\lambda \geq 0,$$
where λ is an affine parametrization and k^μ is the corresponding tangent vector.

6. The averaged weak energy condition (AWEC) - this condition is said to hold on a timelike curve Γ if:
$$\int_\Gamma T_{\mu\nu}V^\mu V^\nu ds \geq 0,$$
where s is the proper time parametrization of the timelike curve Γ and V^μ is the corresponding tangent vector.

7. The averaged strong energy condition (ASEC) - this condition is true for a timelike curve Γ if:

$$\int_\Gamma \left(T_{\mu\nu} V^\mu V^\nu + \frac{1}{2} T \right) ds \geq 0$$

At the end of section 3 we stated Morris's and Thorne's condition that in a traversable wormhole's throat the tension must be larger than the total density of mass-energy $\tau_0 > \rho_0$ - the so-called exotic material property. The implication for this was that an observer falling through the wormhole with a radial velocity close to the speed of light will measure a negative mass-energy density - with the natural logical extension for a static observer measuring a negative mass-energy density. This contradicts multiple energy conditions mentioned above in which the mass-energy density was strictly non-negative, such as the weak energy condition and the dominant energy condition.

4.2 Raychaudhuri-Landau equation

A fundamental result for the motion of adjacent particles was discovered independently by the Indian physicist Amal Kumar Raychaudhuri [4] and by the Soviet physicist Lev Landau [15]. This result is referred to as the Raychaudhuri-Landau equation, or sometimes only as Raychaudhuri's equation. Loosely speaking, this equation describes the congruence or divergence of a family of world lines at specific spacetime locations. More precisely, it characterizes the rate of change of expansion of a family of timelike or null curves, which could represent flow lines or histories of photons [16].

The shape of this equation is:

$$\dot{\theta} = -\frac{\theta^2}{3} - 2\sigma^2 + 2\omega^2 - E[\boldsymbol{X}]^a_a + \dot{X}^a_{;a},$$

where $\boldsymbol{X}$ is a timelike unit vector field and $E[\boldsymbol{X}]^a_a = R_{mn}X^mX^n$ (sometimes called the Raychaudhuri scalar) is the trace of the tidal tensor $E[\boldsymbol{X}]_{ab} = R_{ambn}X^mX^n$. $2\sigma^2 = \sigma_{mn}\sigma^{mn}$ and $2\omega^2 = \omega_{mn}\omega^{mn}$ are non-negative quadratic invariants of the shear tensor $\sigma_{ab} = \theta_{ab} - \frac{1}{3}\theta h_{ab}$ and the vorticity tensor $\omega_{ab} = h^m_a h^n_b X_{[m;n]}$, respectively. In these expressions, $\theta_{ab} = h^m_a h^n_b X_{(m;n)}$ is the expansion tensor, θ is its trace, called the expansion scalar, and $h_{ab} = g_{ab} + X_a X_b$ is the projection tensor onto the hyperplanes orthogonal to $\boldsymbol{X}$.

There are two different types of terms present in this equation:

1. Terms which encourage collapse:

   - initial nonzero trace of the expansion tensor θ_{ab} (nonzero expansion scalar θ);
   - nonzero shearing σ^2;
   - positive trace of the tidal tensor (Raychaudhuri scalar) $E[\boldsymbol{X}]^a_a = R_{mn}X^mX^n$ - the strong energy condition implies exactly this.

2. Terms which oppose collapse:

   - nonzero vorticity ω^2, corresponding to Newtonian centrifugal forces;
   - positive divergence of the acceleration vector $\dot{X}^a_{;a}$.

For the purpose of our paper, we will analyze the situation when $\boldsymbol{X}$ is a timelike geodesic unit vector field with vanishing vorticity or which is hypersurface orthogonal - the mathematics are the same. This could be the case for a dust particle travelling along a timelike geodesic which solves Einstein's field equations - assuming that the world lines do not twist around each other. More than that, we assume the validity of the strong energy condition (which is expected to be true for reasonable classical matter). In this case, Raychaudhuri's equation becomes:

$$\dot{\theta} = -\frac{\theta^2}{3} - 2\sigma^2 - E[\boldsymbol{X}]^a_a$$

The right hand side of the equation will always be negative or zero, which means that the expansion scalar θ does not increase with time. The second and third term are always negative, thus

$$\dot{\theta} \leq -\frac{\theta^2}{3}$$

We can integrate this inequality with respect to the proper time τ:

$$\int_{\theta_0}^{\theta_{final}} \frac{d\theta}{\theta^2} \leq \int_0^{\tau_{final}} -\frac{1}{3} d\tau$$

$$-\theta_{final}^{-1} - (-\theta_0^{-1}) \leq -\frac{1}{3}\tau_{final}$$

$$\frac{1}{\theta_0} - \frac{1}{\theta_{final}} \leq -\frac{1}{3}\tau_{final}$$

$$\frac{1}{\theta_{final}} \geq \frac{1}{3}\tau_{final} + \frac{1}{\theta_0}$$

If the initial value θ_0 is negative, the value for θ_{final} can reach minus infinity in a finite proper time, with a value at most equal to $\frac{3}{\theta_0}$. This is something known as the focusing theorem: If the strong energy condition (SEC) holds and the geodesic congruence is timelike (a family of free-falling particles), all geodesics leaving a point will eventually reconverge after a finite time. The equivalent is true for the null energy condition (NEC) with null geodesic congruence (a family of freely propagating light rays) [17]. This result has been often used by Hawking and Penrose in their various formulations of the Penrose–Hawking singularity theorem [16], [17].

The existence of a traversable wormhole is possible only if the geodesics entering the wormhole on one side (and thus converging as they approach the throat) will emerge on the other side diverging away from each other. By Raychaudhuri's equation, this can only happen if certain energy conditions are violated - the null energy condition (NEC) and the averaged null energy conditions (ANEC) are of particular interest. The discussion above serves as an intuitive way to explain why this is the case, and why exotic matter or configurations which violate energy conditions are needed for a traversable wormhole to exist.

A more rigorous discussion about the necessity of NEC violations was published in 1995 by Friedman, Schleich and Witt, under the name of “Topological censorship”. They summarized this constraint as follows ”General relativity does not allow an observer to probe the topology of spacetime: any topological structure collapses too quickly to allow light to traverse it” [18]. This condition was extended to anti-de Sitter spacetime in 1999 by Galloway, Schleich, Witt, and Woolgar [19]. Thus, traversable wormholes would not exist under these circumstances.

4.3 ANEC arguments

The weakest energy condition of those presented above is the averaged null energy condition (equation 5 in section 4.1). The whole discussion up until this point focused on the necessity to violate at least this condition, if not a stronger one, to obtain traversable wormholes, and on the assumption that this condition is true for regular matter-energy. However, even if this assumption is believed to be true, it has not been proven for all topologies and metrics. In this section we will look at some particular cases in which this constraint is satisfied and the logical extensions to other cases.

In Minkowski space, the averaged null energy condition has been proven for free scalar fields in any dimension [20], free electromagnetic fields in four dimensions [21], and for any QFT with a mass gap in two-dimensions [22]. More recently, the ANEC has also been proven in the Minkowski spacetime by using the monotonicity of relative entropy and modular Hamiltonians in relativistic quantum field theories on $\mathbb{R}^{1,d-1}$ [23] and by using unitary, Lorentz-invariant quantum field theories in flat spacetime whose commutators vanish at spacelike separation - microcausality. This characteristic implies ANEC for interacting theories in more than two dimensions [24].

A more general approach provides a proof of ANEC based on the Generalized Second Law (GSL), an extension of the ordinary second law of thermodynamics (OSL). OSL states that the total thermodynamic entropy of the Universe is always increasing with time. GSL states that the total generalized entropy is never decreasing with time, where the expression for generalized entropy is given by:

$$\frac{kA}{4G\hbar} + S_{out},$$

where k is the Boltzmann constant, G is the gravitational constant, $\hbar$ is Planck's constant divided by 2π, the speed of light c is considered to be 1, A is the sum of the areas of all black holes in the Universe, and S_{out} is the ordinary thermodynamic entropy of the system outside all event horizons. This relation was theorized by Bekenstein and it is based on the similar behaviour of blackhole areas and of entropy [25]. This law has so far not been proved, but many attempts are constantly being made towards achieving this goal [26]. It is widely believed that GSL is true and that it will be eventually proven. In a 2010 paper by Wall, ANEC is proved to be true starting from the assumption that GSL is true [27].

Given the constantly increasing number of proofs of ANEC in specific environments, and its general proof starting from GSL, the averaged null energy condition is expected to hold for reasonable classical matter or quantum fields on asymptotically flat or flat spacetimes.

4.4 Classical case

Wormholes do exist in the classical case - for example, the Einstein-Rosen bridge. By "the classical case" we understand the laws of physics as described by general relativity and electromagnetism, without considering any quantum effects. So far three different arguments have been presented to support that a traversable wormhole is impossible without ANEC violation: the need for exotic matter with $\tau_0 > \rho_0$ [14], the Raychaudhuri-Landau equation [4], and the topological censorship [18]. It is expected that the null energy condition (and thus ANEC) holds for classical matter - as discussed in the section above, so flat or asymptotically flat spacetimes coupled only with classical matter are not environments where traversable wormholes could occur.

4.5 Quantum case

The averaged null energy condition is violated by quantum fields on curved spacetimes, giving rise to negative energy densities. As opposed to the classical case, this brings about the possibility of traversable wormholes. Such violations can appear in various ways:

- Negative mass matter - a purely theoretical model.
- String theory comes as a solution to the ANEC violation problem. For example, negative tension branes violate all the standard energy conditions of the higher-dimensional spacetime they are embedded in [28].
- The topological Casimir effect, which occurs, for example, in a universe with a periodicity condition in one of the directons [1].
- Casimir-like systems in Minkowski space in which one spatial dimension has been compactified [29]. In this case, both the energy density and the pressure

in the compactified direction are negative everywhere. ANEC is violated along geodesics going in the compact direction. This will be the method used in the main paper detailed in this work.

- Boulware vacuum - Matt Visser published a paper which shows that the averaged energy conditions are violated throughout the whole exterior of a black hole - from spatial infinity to the event horizon - in this environment [30].
- Hartle-Hawking vacuum - the ANEC is violated for circular curves existing between the event horizon of the Schwarzschild black hole and the $r = 3M$ unstable circular photon orbit [1], [31].
- Urban and Olum found a violation of ANEC through a conformal transformation of a spacetime (in the paper the Minkowski spacetime was considered) which obeys ANEC but violates NEC. This violation can happen both anomalously and in a nonanomalous way [29].
- The interaction between two conformal filed theories on the boundaries of an eternal BTZ black hole spacetime induces a negative energy density, and thus a violation of ANEC [32].
- Unruh vacuum and evaporating black holes - Ford and Roman described situations in which ANEC is violated in this context: in 4D, it is violated for outgoing null geodesics and orbiting null geodesics, while in 2D it is violated for half-complete outgoing radial null geodesics and orbiting null geodesics on the horizon. More than that, they bound the magnitude of the negative energy density through "quantum inequalities" [33].

These are only a few examples where the averaged null energy density condition is violated. In the quantum case, ANEC violation is a phenomenon which is achievable through multiple means. The idea of a traversable wormhole is no longer pure fiction, and in section 5 of this paper we will actually explore some specific constructions of such wormholes, using one of the examples above to create a negative energy density.

4.6 Time travel

The concept of wormholes is inevitably linked to the idea of travel backwards or forwards through time. Time travel is described in a mathematical way through the existence of closed timelike curves (CTCs) - these are worldlines which material particles can use to move through spacetime and to arrive back at the same point. They were first theorized by Willem Jacob van Stockum in 1937 [34], and later on they were confirmed by Kurt Gödel in 1949 [35]. Gödel discovered a new solution to Einstein's field equations where the stress-energy tensor has two components: a matter-density component of a homogeneous distribution of dust particles and a nonzero cosmological constant one. The metric is of the form:

$$ds^2 = a^2 \left(dx_0^2 - dx_1^2 + \frac{e^{2x_1}}{2} dx_2^2 - dx_3^2 + 2e^{x_1} dx_0 dx_2 \right)$$

Solving Einstein's field equations with this metric, we obtain some values for the cosmological constant and the dust density [35]:

$$\Lambda = -\frac{1}{2a^2} \qquad \rho = \frac{1}{8\pi a^2}$$

As we see, both of them depend on the value of the constant a and are thus not independent of each other. There is no physical reason for the Universe to evolve in a way which would relate the cosmological constant to the quantity of existing matter. This is why this solution is sometimes called “artificial”. This metric has a few other surprising characteristics, besides the existence of CTCs. It is a rare example of a singularity-free solution of the Einstein field equations. Furthermore, it is a cosmological model of a rotating universe (angular velocity for all matter $\frac{1}{\sqrt{2}a}$) with no Hubble expansion (and thus no cosmological redshift) - a steady state universe [35].

Since Gödel's discovery, scientists have tried to prove or disprove the existence of such curves given our current understanding of physics, and to understand whether they could be somehow created in our universe. One of the first papers which comes to mind was published by Tipler in 1976. He concludes that it is impossible to create CTCs using only reasonable classical matter in a singularity free asymptotically flat spacetime [36]. Hawking was another scientist deeply preoccupied by causality questions. In 1992 he published a paper in which he theorized and provided supporting arguments to what he called the " chronology protection conjecture: The laws of physics do not allow the appearance of closed timelike curves" [37]. In the classical case, CTCs are precluded by the need to violate the averaged weak energy condition (AWEC). In the quantum case (where such a violation is no longer an issue), Hawking states that the expectation value of the stress-energy tensor would become infinitely large when timelike curves become almost closed, which would prevent closed timelike curves from appearing [37]. Thus, time travel through traversable wormholes is not possible.

More than that, this helps explain why short traversable wormholes are forbidden. Let's analyze the case where a short wormhole has a zero length throat and a small enough radius to consider two different worldlines completely identified with one another. If one end of the wormhole was boosted with a relativistic velocity, and then boosted back to its initial position, a time delay would occur between the two clocks associated to the two different mouths - similar to what happens in the twin paradox. If the time delay is sufficiently large compared to the distance between the two ends of the wormhole through the regular spacetime, a time machine is created - an observer could now travel along closed timelike curves [13] - figure 4.1. In the paragraph above, this possibility was excluded; thus short traversable wormholes cannot exist.

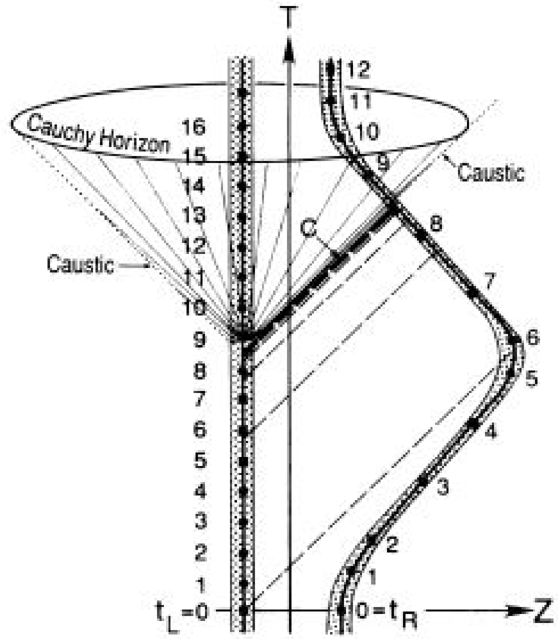

Figure 4.1: Creating a time machine by boosting one of the mouths of the traversable wormhole. Figure taken from [13].

Another way to go about proving the impossibility of short traversable wormholes is the self-consistent achronal averaged null energy condition (SCAANEC). This condition states that “There is no self-consistent solution in semiclassical gravity in which ANEC is violated on a complete, achronal null geodesic.” [38]. An achronal geodesic is one which doesn’t contain any points connected by a timelike path. This is a weaker energy condition than ANEC. It is strong enough to rule out exotic phenomena as effectively as ordinary ANEC, but weak enough to avoid known violations of ANEC. Graham and Olum provide arguments in their 2007 paper about the validity of this new energy condition, concluding the paper by stating that this condition is sufficient to rule out CTCs (thus ruling out short traversable wormholes) and wormholes connecting different asymptotically flat regions [38]. Note: in a 2010 paper by Urban and Olum, two violations of the achronal averaged null energy condition (AANEC) have been described. The first was obtained by a transformation which amplifies the NEC-violating portions of a sequence of excited states. The second violation was constructed purely from the geometric anomalous terms in the stress tensor. Both of these violations can become arbitrarily large [29]. However, the addition of the self-consistency criteria could be one possibility to exclude exotic phenomena from general relativity.

Section 5

Constructions

5.1 Traversable wormhole with fermions

This section of the dissertation will focus on the paper published in 2018 by Maldacena, Milekhin and Popov, in which they explicitly describe a plausible traversable wormhole solution in four dimensions [39]. The theory considered is a solution of the Einstein-Maxwell equations with the addition of a U(1) gauge field coupled to a set of massless Weyl fermions. Violation of the averaged null energy condition (ANEC) - a mandatory setting for the existence of traversable wormholes (as discussed in section 4) - is achieved here through a negative Casimir-like energy produced by the charged fermions. The solution can also be regarded as "a pair of entangled near extremal black holes with an interaction term generated by the exchange of fermion fields" [39]. For this reason, we will start the analysis by looking at magnetically charged black holes, their interactions with charged fermions, and the negative Casimir-like energy resulting from these interactions, while the actual construction of the wormhole, its metric and its characteristics will follow afterwards.

5.1.1 Single magnetic black hole geometry

For a magnetically charged black hole, the metric will be:

$$ds^2 = -\left(1 - \frac{2MG}{r} + \frac{r_e^2}{r^2}\right) dt^2 + \left(1 - \frac{2MG}{r} + \frac{r_e^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2),$$

where $r_e^2 = \frac{\pi q^2 G}{g^2}$, M is the mass of the black hole, q is an integer representing the charge of the black hole, and g is the coupling constant of the U(1) gauge field. The horizon is localized at $r = r_+$, where

$$r_\pm = MG \pm \sqrt{M^2G^2 - r_e^2}$$

The relations for temperature and entropy are:

$$T = \frac{r_+ - r_-}{4\pi r_+^2} \qquad S = \frac{\pi r_+^2}{G}$$

The extreme points are reached when $T \to 0$ and $r_+ = r_- = r_e$. There, M and S can be expanded around a small T to obtain:

$$M = \frac{r_e}{G} + \frac{2\pi^2 r_e^3 T^2}{G} + \dots$$

$$S = \frac{\pi r_e^2}{G^2} + \frac{4\pi^2 r_e^3 T}{G^2} + \dots$$

The $AdS_2 \times S^2$ geometry approximates really well the geometry near the horizon:

$$ds^2 = r_e^2 \left[-d\tau_r^2(\rho_r^2 - 1) + \frac{d\rho_r^2}{\rho_r^2 - 1} + (d\theta^2 + sin^2\theta d\phi^2) \right],$$

where $\tau_r = 2\pi Tt$, $\rho_r = \frac{r - r_e}{2\pi T r_e^2}$ for $r - r_e << r_e$. This metric is a good approximation when $r - r_e << r_e$ and $Tr_e << 1$. This geometry connects with the flat spacetime through a transition region located around $r \approx r_e$ (Figure 5.1).

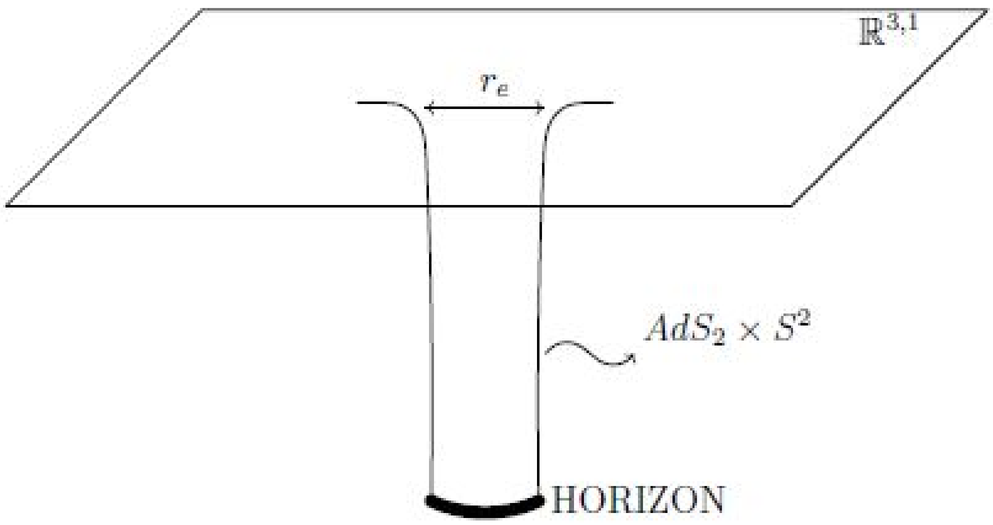

Figure 5.1: A drawing representing a near extremal black hole geometry. The throat has an $AdS_2 \times S^2$ geometry. Image taken from [39].

5.1.2 Fermion dynamics

Now let us take a look at the behavior of a single charged Dirac fermion with charge $Q = 1$. The action in this case is:

$$I = \int d^4x\sqrt{g}\left[\frac{R}{16\pi G} - \frac{1}{4g^2}F^2 + i\bar{\chi}(\not\nabla - i\not A)\chi\right],$$

where the vector potential A is given by the magnetically charged black hole:

$$A = \frac{q}{2}cos\theta d\phi$$

In the presence of this magnetic field, the massless charged fermion gives rise to a series of Landau levels. The energy of a Landau level has an orbital contribution and a magnetic dipole contribution. For the lowest Landau level in the fermion's case, these two contributions cancel exactly, giving rise to a zero-energy state [40]. This state has a degeneracy q, which is related to the corresponding angular momentum j through the following relation: $2j+1 = q$. Thus, a four dimensional chiral fermion gives rise to q massless two dimensional chiral fermions - a crucial idea for this construction [39].

We can factorize the four dimensional spinor as:

$$\chi = \psi \otimes \eta,$$

where η is a spinor on the S^2 sphere and ψ is a spinor in the other two directions. The lowest Landau level corresponds to a negative chirality spinor η_- that obeys the two dimensional massless Dirac equation on the two sphere with a magnetic field

$$\gamma^\alpha(\nabla_\alpha - iA_\alpha\eta) = 0$$

The solutions to these equations have the form:

$$\eta_- \propto e^{im\phi}\left(sin\frac{\theta}{2}\right)^{j-m}\left(cos\frac{\theta}{2}\right)^{j+m},$$

where $j = \frac{q-1}{2}$ and $-j \leq m \leq j$. For $q >> 1$, these solutions are well localized around $\theta_m = \frac{m}{j}$. Each of these modes on the sphere gives rise to a two dimensional massless mode Ψ_m on the r and t directions. These equivalent fermions propagate through space with the metric:

$$ds_2^2 = |g_{tt}|(-dt^2 + dx^2),$$

where $dx = \sqrt{\frac{g_{rr}}{-gtt}}dr$. They are the creators of a negative Casimir-like energy. If a scalar field were to be considered instead of a fermionic field, all Landau levels energies would be positive. These fields decay rapidly in the throat region and no significant Casimir energy can be obtained this way.

5.1.3 Two interacting magnetic black holes

When seen from a large distance d, where $d >> r_e$, two oppositely charged black holes will behave as a magnetic dipole. The vector potential created by this configuration will then be:

$$A = \frac{q}{2}(cos\theta_1 - cos\theta_2)d\phi,$$

where θ_1 and θ_2 are the angles represented in figure 5.2.

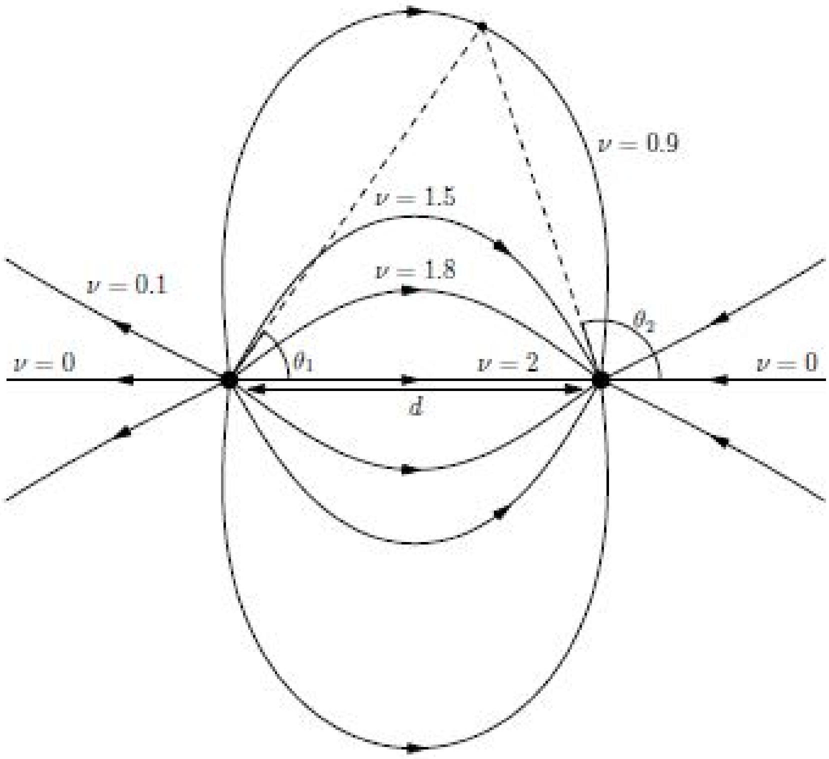

Figure 5.2: Magnetic sources at distance d from each other and their field lines. Figure taken from [39].

The configuration is rotational invariant around the axis connecting the two sources. We consider this to be the z axis, and we define the angle around it to be ϕ. Using cylindrical coordinates, the spatial metric will be:

$$ds^2 = d\rho^2 + dz^2 + \rho^2 d\phi^2$$

In these coordinates, the components of the magnetic field are:

$$B_\rho = \frac{\partial_z A_\phi}{\rho} \qquad B_z = -\frac{\partial_\rho A_\phi}{\rho} \qquad B_\phi = 0$$

The tangent vector along the field lines has to be along $\boldsymbol{B}$. If the solution $A_\phi = constant$ is considered, then its gradient will be normal to $\boldsymbol{B}$ and the condition above will be fulfilled. The equations of the field lines will then be:

$$cos\theta_1 - cos\theta_2 = \nu \qquad 0 \leq \nu \leq 2 \qquad \nu = \frac{j+m}{j},$$

where we connected the geometry of the magnetic field lines to the fermion dynamics described in the previous section.

The length of the trajectory traveled by a fermion along a field line will be dependent on ν and the distance d between the two mouths:

$$L_{fieldline} = df(\nu),$$

where the shape of the function $f(\nu)$ can be determined through a coordinate transformation from z and ρ to θ_1 and θ_2 [39].

5.1.4 Wormhole assembly

Now we can put all these elements together to describe the resulting traversable wormhole created by connecting the throats of the two magnetically charged black holes. The paper splits the geometric configuration into three different regions: the actual wormhole (red), the two mouths (blue), and the flat space around them (green) (figure 5.3).

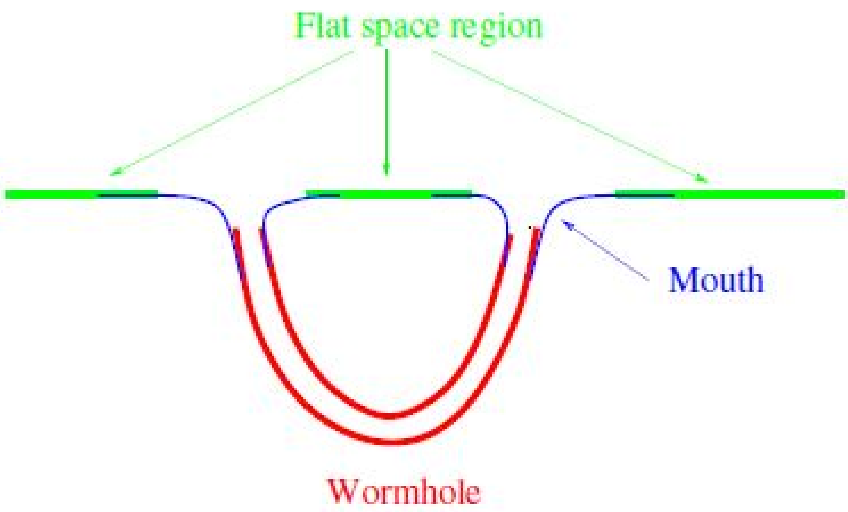

Figure 5.3: The three regions in which the wormhole solution is split. The individual metrics coincide in the overlapping transition regions. Image taken from [39].

The actual wormhole (red part) will be characterized by a metric similar to that of the throat of a near extremal black hole - after all, the wormhole was obtained by connecting two such throats. As we saw in part 5.1.1, this metric is well approximated (at a first level) by the $AdS_2 \times S^2$ geometry:

$$ds^2 = r_e^2 \left[-d\tau^2(\rho^2+1) + \frac{d\rho^2}{\rho^2+1} + (d\theta^2 + sin^2\theta d\phi^2) \right]$$

Note: these coordinates are different than the ones used in section 5.1.1. - this is the "global" coordinate system which covers the whole Penrose diagram of AdS_2. The asymptotic regions of this metric ($\rho >> 1$) will be matched to the metric of the magnetically charged extremal black hole ($r - r_e << r_e$) - this overlap corresponds to the red-blue transition zone on the diagram in figure 5.3. The match will happen via the following equalities:

$$\tau = \frac{t}{l} \qquad\qquad \rho = \frac{l(r - r_e)}{r_e^2},$$

where l is a length-like free parameter which provides a rescaling between the AdS_2

time and the asymptotic time (similar to the inverse temperature $\frac{1}{T}$ in section 5.1.1.). Its value will be fixed later on. Given the two conditions for the overlapping section: 1) $\rho >> 1$ from the wormhole metric and 2) $r - r_e << r_e$ from the mouth metric, we can find an inequality which involves l:

$$\rho = \frac{l(r - r_e)}{r_e^2} >> 1 \quad \& \quad \frac{r - r_e}{r_e} << 1 \quad \Rightarrow \quad \frac{l}{r_e} >> 1$$

Considering that the two metrics are different and only approximately equivalent in a specific area, there will exist a cutoff point past which the overlap will not make sense anymore. The throat opens up around $r - r_e \approx r_e$, so it is natural to consider the cutoff radius will be proportional to $\frac{l}{r_e} >> 1$.

We can calculate the rescaled length of the wormhole in the coordinates introduced at the end of section 5.1.2., in the limit $\rho_{cutoff} \to \infty$:

$$L_{throat} = \int_{-\rho_{cutoff}}^{\rho_{cutoff}} d\rho \frac{l}{\rho^2 + 1}$$

$$L_{throat} = l \left[arctan(\rho_{cutoff}) - arctan(-\rho_{cutoff}) \right]$$

$$L_{throat} = l \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = l\pi$$

Next, in order to be able to calculate the negative energy density, we need to calculate the length of a fermion's trajectory. These two-dimensional fermions exist on circular curves: they exit through one mouth of the wormhole, travel along a magnetic field line until they reach the other mouth, they enter the second mouth and then they travel through the wormhole until they once again reach the original mouth. We can write the whole length of the curve as the sum between the length of the wormhole, the length of the field line and two small transition segments around the mouths (these small lengths will be considered negligible). More than that, we will choose to analyze the system using the approximation $d << l$ - the length of the throat is much larger than the distance between the mouths of the wormhole. In this case, the length of the trajectory is simply the length of the throat $L_{throat} = l\pi$.

The Casimir-like energy is composed of two terms: the ground state contribution of the fermions moving on the circle of length L and a contribution due to the two dimensional conformal anomaly - in the wormhole region the spacetime in which the fermions live is not flat, it is only conformally flat:

$$E_{wormhole} = E_{ground\ state} + E_{conformal\ anomaly}$$

For q fermions moving along a circle of length L this energy will be:

$$E_{wormhole} = -\frac{q}{12}\frac{2\pi}{L} + \frac{q}{24}\frac{\pi}{L}$$

$$E_{wormhole} = -\frac{q}{24}\frac{3\pi}{L} = -\frac{q}{8}\frac{\pi}{\pi l} = -\frac{q}{8l}$$

The semiclassical Einstein equations can be solved directly now. We need to consider two different regions: one corresponding to the asymptotically flat spacetime outside the wormhole, and one corresponding to the throat region. The two solutions have to overlap at the mouth region. A spherically symmetric ansatz is used for the metric in the asimptotic case:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2$$

The most general solution has only one integration parameter - the mass M - which will be set later on by the overlap condition. We obtain the result:

$$A = \frac{1}{B} = \left(1 - \frac{r}{r_e}\right)^2 - \frac{2\epsilon}{r_e},$$

where $\epsilon = GM - r_e$.

In the wormhole region, we expand the metric perturbatively away from $AdS_2 \times S^2$ metric:

$$ds^2 = r_e^2 \left(-(1+\rho^2+\gamma)d\tau^2 + \frac{d\rho^2}{1+\rho^2+\gamma} + (1+\phi)(d\theta^2 + sin^2\theta d\phi^2 \right),$$

where γ and ϕ are small.

The electromagnetic tensor in this situation is:

$$F = dA = -\frac{q}{2} sin\theta d\theta d\phi$$

The quantum contributions to the stress-energy tensor can be written as:

$$\hat{T}_{tt} = \hat{T}_{xx} = -\frac{q}{8\pi l^2}\frac{1}{4\pi r_e^2}$$

$$\hat{T}_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T_\epsilon^\epsilon, \quad where \quad \alpha, \beta = t, x.$$

The four dimensional stress tensor also contains a classical contribution from the magnetic field. The $\rho\rho$ component of Einstein's field equation gives us:

$$R_{\rho\rho} - \frac{1}{2} g_{\rho\rho} R - 8\pi G \left(T_{\rho\rho}^{mag} + \hat{T}_{\rho\rho} \right) = 0$$

We can calculate $\hat{T}_{\rho\rho}$ from:

$$\hat{T}_{\rho\rho} = \frac{\hat{T}_{\tau\tau}}{(1+\rho^2)^2} = \frac{l^2 \hat{T}_{tt}}{(1+\rho^2)^2}$$

$$\hat{T}_{\rho\rho} = -\frac{l^2 q}{8\pi l^2}\frac{1}{4\pi r_e^2}\frac{1}{(1+\rho^2)^2} = -\frac{q}{8\pi}\frac{1}{4\pi r_e^2 (1+\rho^2)^2}$$

Then from Einstein's equation mentioned above we obtain:

$$\rho\phi' - \phi = (8\pi G)(1+\rho^2)\hat{T}_{\rho\rho}$$

$$\rho\phi' - \phi = -(8\pi G)(1+\rho^2)\frac{q}{8\pi}\frac{1}{4\pi r_e^2(1+\rho^2)^2}$$

$$\rho\phi' - \phi = -G\frac{q}{4\pi r_e^2(1+\rho^2)}$$

For simplicity, the following notation will be used:

$$\rho\phi' - \phi = -\frac{\alpha}{(1+\rho^2)}, \qquad \alpha = \frac{qG}{4\pi r_e^2}$$

Integrating the ecuation we get:

$$\phi = \alpha(1 + \rho\, arctan(\rho))$$

In doing so, the additional term linear in ρ resulting from the integral was set to 0, since our system is symmetric around $\rho = 0$.

We move on to overlapping the two metrics and identifying the corresponding coefficients. Notice that we are working in the $\rho >> 1$ limit, which is equivalent to considering $\frac{r-r_e}{r_e} << 1$. By matching the sphere part of the two metrics we obtain:

$$r^2 = r_e^2(1+\phi) \Rightarrow 1+\phi = \frac{r^2}{r_e^2}$$

$$\phi = \frac{r^2}{r_e^2} - 1 = \frac{r^2 - r_e^2}{r_e^2} = \frac{(r-r_e)(r_e+r)}{r_e^2}$$

The difference between r and r_e is very small in this limit, so we can approximate their sum to be $r + r_e \approx 2r_e$. Then ϕ becomes:

$$\phi \approx \frac{2(r-r_e)}{r_e}$$

We can also calculate ϕ by approximating the solution obtained through integration in the large ρ limit:

$$\phi = \alpha(1 + \rho\ arctan(\rho)) \approx \alpha\rho\ arctan(\rho) \approx \alpha\rho\frac{\pi}{2}$$

From these two equalities we can find an expression for ρ:

$$\rho = \frac{4(r - r_e)}{r_e\alpha\ \pi}$$

We can now compare the leading order time component to find a value for l:

$$\left(1 - \frac{r}{r_e}\right) dt = r_e\rho d\tau \Rightarrow$$

$$\Rightarrow \left(1 - \frac{r}{r_e}\right) dt = r_e\frac{4(r - r_e)}{r_e\alpha\ \pi}\frac{dt}{l}$$

$$\Rightarrow \frac{1}{r_e} = \frac{4}{\alpha\ \pi}\frac{1}{l} \Rightarrow$$

$$\Rightarrow l = \frac{4r_e}{\alpha\ \pi} = \frac{4r_e}{\pi}\frac{4\pi r_e^2}{qG} \Rightarrow$$

$$\Rightarrow l = \frac{16r_e^3}{qG}$$

The energy of the configuration is (energy of two near extremal black holes + the Casimir-like contribution):

$$E = \frac{{r_e}^3}{l^2 G} + E_{wormhole} = \frac{{r_e}^3}{l^2 G} - \frac{q}{8l}$$

By using the expression for l obtained above, we can calculate this energy as:

$$E = \frac{{r_e}^3}{G}\frac{q^2 G^2}{256 r_e^6} - \frac{q}{8}\frac{qG}{16 r_e^3}$$

$$E = \frac{q^2 G}{r_e^3}\left(\frac{1}{256} - \frac{1}{128}\right) = -\frac{q^2 G}{256 r_e^3}$$

Another way to express this energy is by using the mass correction ϵ:

$$E = \frac{2\epsilon}{G},$$

where the factor of 2 comes form the mass correction to each of the two black holes. Matching the exact time component of the two metrics, we can determine the value of the ϵ mass correction:

$$\left(\left(1 - \frac{r}{r_e}\right)^2 - \frac{2\epsilon}{r_e}\right) dt^2 = r_e^2(1+\rho^2)d\tau^2 \Rightarrow$$

$$\Rightarrow -\frac{2\epsilon}{r_e}dt^2 = r_e^2 d\tau^2$$

$$-\frac{2\epsilon}{r_e}dt^2 = r_e^2\frac{dt^2}{l^2} \Rightarrow$$

$$\Rightarrow \epsilon = -\frac{r_e^3}{2}\frac{q^2G^2}{256r_e^6} = -\frac{1}{2}\frac{q^2G^2}{256r_e^3}$$

The energy will then be:

$$E = \frac{2\epsilon}{G} = -\frac{2}{G}\frac{1}{2}\frac{q^2G^2}{256r_e^3} = -\frac{q^2G}{256r_e^3},$$

which is exactly what was obtained using the first method.

Throughout the whole calculation we considered that $d << l$ - the length of the throat is much larger than the distance between the two mouths of the wormhole. This means that the calculation above is valid only when $d << \frac{16r_e^3}{qG}$. But what if this is not the case? Then the total length L of the fermion's trajectory will be:

$$L = \pi l + df(\nu)$$

and the energy has a more complicated expression depending on the particular angular momentum of the charged fermion wavefunction:

$$E = \frac{{r_e}^3}{l^2 G} + \frac{q}{24l} - \frac{q\pi}{6} \int_0^2 \frac{d\nu}{2} \frac{1}{(l\pi + df(\nu))}$$

The first term is the classical energy (same value as before), the second term comes from the conformal anomaly in the throat region (same expression as before), and the last term is the integral over all the magnetic field lines of the Casimir energy (different expression). In order to be able to draw some physical conclusions, we will approximate the integral by $1/(\pi l + d)$ - a constant given by its largest term. Physically, this corresponds to all fermions traveling from one mouth to the other following the shortest path. The energy has the value:

$$E = \frac{{r_e}^3}{l^2 G} + \frac{q}{24l} - \frac{q\pi}{6} \frac{1}{(\pi l + d)}$$

We can find l by minimizing this energy:

$$\frac{\partial E}{\partial l} = 0 \Rightarrow -\frac{2r_e^3}{l^3 G} - \frac{q}{24l^2} + \frac{q\pi}{6} \frac{1}{(\pi l + d)^2} \pi = 0$$

$$\frac{2r_e^3}{l^3 G} = -\frac{q}{6} \left(\frac{1}{4l^2} - \frac{1}{\left(l + \frac{d}{\pi}\right)^2} \right)$$

Another approximation can be used now, where we ignore the left hand side of the equation. In this case, the results for the length l and the minimal energy E will be:

$$4l^2 = \left(l + \frac{d}{\pi}\right)^2 \Rightarrow$$

$$\Rightarrow l = \frac{d}{\pi}$$

$$E = \frac{q\pi}{24d} - \frac{q}{6} \frac{1}{(\frac{d}{\pi} + \frac{d}{\pi})} = \frac{q\pi}{6d} \left(\frac{1}{4} - \frac{1}{2} \right)$$

$$E = -\frac{q\pi}{24d}$$

This is an inferior limit of l given these approximations. Even if we use this value for l, when we calculate the length of the throat we obtain $L_{throat\ min} = d$. Thus, the minimum length of the throat will be equal to the distance between the mouths, confirming our assumption that this has to be a long wormhole.

We see that all the models we used give rise to negative energy densities. These negative energies densities are the key to creating the traversable wormhole.

5.1.5 Stabilizing the two wormhole mouths

From a large distance $r >> r_e$, the two mouths of the wormhole are seen as two extremal black holes with opposite magnetic charges. These would attract each other and eventually collide if no other mechanism prevents it. One way to stabilize the system is to consider the two black holes to be rotating around the common center of mass in a flat spacetime. The angular velocity of this rotation can be calculated from Kepler's third law, while taking into consideration that the attractive centripetal force is composed of a gravitational component and a magnetic component. Its expression is:

$$\Omega = 2\sqrt{\frac{r_e}{d^3}}$$

A small orbital eccentricity would result in small perturbations of the throat with a frequency which is small enough to not destroy the wormhole.

There are a few concerns which occur when the mouths rotate:

- The fermions are affected by the rotation and they feel additional forces (such as the Coriolis force), which disturb their calculated trajectories;
- Radiation needs to be emitted since charged particles are accelerating;
- An Unruh-like temperature (and thus an emission of energy) will occur and it will make the energy inside the throat less negative.

Approximate values for these effects are calculated in Maldacena's, Milekhin's and Popov's paper and neither one seems to be powerful enough to collapse the wormhole [39].

Another way to stabilize the system is to utilize AdS_4 spacetimes. Here, the separating mechanism can be either the rotation of the two black holes around each other in the same AdS_4 spacetime, a specific configuration of the two black holes relative to a magnetic field, or the coupling of two AdS_4 spaces (each containing one of the mouths) through boundary conditions which allow the fermions to go from one space to another. The details of these phenomena are beyond the scope of this thesis.

5.1.6 Discussion

The construction presented above can actually be fitted into the Standard Model of particle physics with Einstein gravity, if the distance d between the two black holes is smaller than the electroweak scale. This is the condition for the fermions to be approximated as being massless. Since there exist an ordering relation between d and r_e ($d >> r_e$) the size of the black holes would be much smaller than the electroweak scale too (microscopic wormholes) - no significant object could pass through the wormhole.

This configuration requires no 'exotic' matter to violate the averaged null energy condition (ANEC). The near extremal magnetic black holes could have a large charge $q >> 1$. The action of the magnetic field turns any one charge 4-dimensional fermion into q 2-dimensional fermions which move on circular trajectories (charged 4-dimensional fermions moving in a magnetic field have a zero energy Landau level on the sphere with large degeneracy proportional to q [39]). The charged fermions moving along circular orbits generate negative Casimir-like energy through quantum effects. Due to the possible large charge of the black holes, a very big number of 2-dimensional equivalent fermions could be obtained, which in turn would produce a large amount of negative energy. This creates the traversable wormhole. In order to maintain it stable for long periods of time, multiple methods are examined, including rotation of the wormhole's mouths around each other and placing the mouths in AdS_4 spacetimes.

5.2 Other recent papers

In the previous section, a possible wormhole solution was presented in detail, starting with its geometric construction, the mechanism employed to violate the averaged null energy condition and possible methods used to stabilize the wormhole. In the following pages, other recent wormhole constructions will be briefly discussed in a chronological order, with the focus on the way ANEC violation is achieved in each case and on the novel ideas each article brought to this field.

5.2.1 Traversable wormholes via a double trace deformation

In 2017 Gao, Jefferis and Wall published a paper discussing a novel construction of a traversable black hole [32]. They begin the set-up with an eternal BTZ black hole (Bañados, Teitelboim and Zanelli) - a black hole solution for $(2+1)$-dimensional topological gravity with a negative cosmological constant [41], which contains a non-traversable Einstein-Rosen bridge. Next, a double trace deformation is turned on between the two Conformal Field Theories (CFTs) living on the boundaries with the same time orientation. The interaction is kept active only for a short period of time. This connection creates a quantum matter stress tensor with a negative average null energy - ANEC violation. A gap opens up between the energy levels in the bulk of the wormhole (E_1 and E_2 in figure 5.4). The future event horizon is modified and shifted upwards (orange curve in figure 5.4). Due to this shift, a throat of size ΔV opens up and allows particles from one side of the black hole to reach the other side in a finite amount of time (pink trajectory in figure 5.4). However, due to the short duration of the phenomenon, only a limited number of particles with the right characteristics will be able to pass through the wormhole. Thus, this particular Einstein-Rosen bridge becomes briefly traversable (it is an eternal wormhole, but not an eternally traversable one).

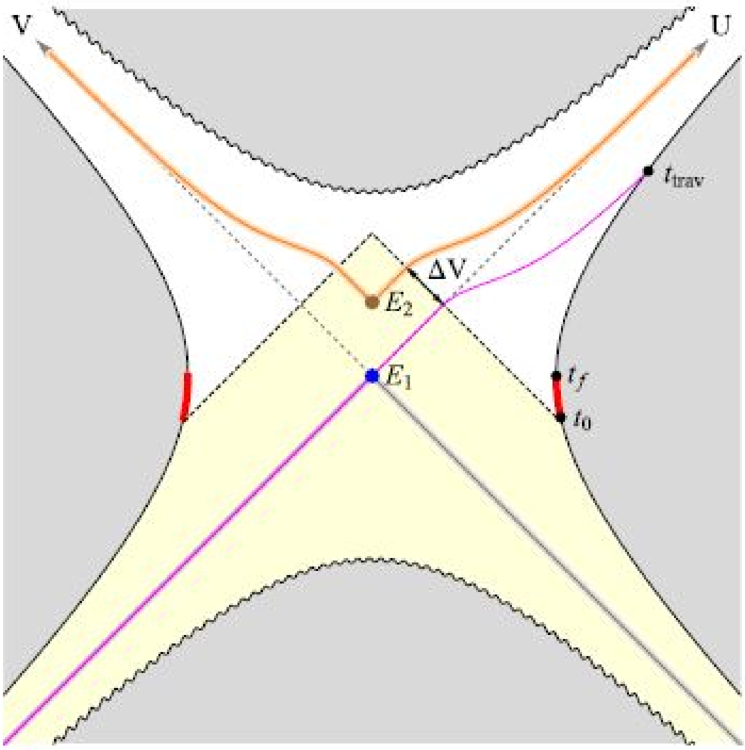

Figure 5.4: The Gao-Jefferis-Wall traversable wormhole construction by direct boundary coupling. The interaction between boundaries begins at t_0 and ends at t_f - the red segment in the figure. The orange curve is the new future event horizon. The grey line is the past event horizon - unmodified by this interaction. The pink trajectory is a null ray which passes through the wormhole. Figure taken from reference [32].

Note that the way the two boundaries are connected together fixes the relative time coordinate between them, excluding the possibility of having closed time-like curves - no time travel is possible in this configuration.

5.2.2 Eternal traversable wormhole

A similar method was used by Maldacena and Qi to construct an eternal traversable wormhole in 2018 [42]. The negative null energy is generated by quantum fields under the influence of an external coupling between the two boundaries. The difference here consist in a time-independent coupling of the boundaries, the time independence being the reason for the eternal characteristic of the wormhole. The construction uses Nearly-$AdS2$ gravity, where all gravitational degrees of freedom live on the boundary. For more details, see reference [42].

5.2.3 Creating a traversable wormhole by a non-perturbative process in quantum gravity

In 2019 Horowitz, Marolf, Santos and Wang published a paper in which they described a traversable wormhole produced through a mechanism not mentioned in this thesis until now. They considered spacetimes with instantons which give a finite probability for a test cosmic string to break and produce two particles on its ends [43]. Instantons are solutions to the equations of motion of the classical field theory on a Euclidean spacetime [44]. If the two particles created by the string are replaced by small black holes, only minimal changes to the spacetime are necessary. These changes will be considered to be negligible.

While in the traditional approach the two particles created this way (and thus the two black holes) accelerate away from each other, in this paper a particular case is considered where they only have a small acceleration, giving them a small oscillatory motion around an equilibrium position (nearly-static particles). Conditions on the spacetime are determined in order to obtain this kind of motion. This is important since wormholes become harder and harder to make traversable as their mouths become more widely separated [43]. The black holes have their horizons identified at the moment of creation, and thus a wormhole is produced. With appropriate boundary conditions, the backreaction (mass and charge corrections to the approximate solution) of quantum fields will render the wormhole traversable in the semiclassical approximation. This is how ANEC violation is obtained in this case.

5.2.4 A perturbative perspective on self-supporting wormholes

In the same year, a similar method based on gravitational backreaction from linear quantum fields was used by Fu, Grado-White and Marolf to turn non-traversable wormholes into traversable ones. The difference here comes from choosing appropriate (periodic or anti-periodic) boundary conditions around a non-contractible cycle, but having natural boundary conditions at infinity (no boundary interactions like the ones in sections 5.2.1 and 5.2.2) [45]. Constructions can be found in asymptotically flat, asymptotically AdS, asymptotically de Sitter spaces or in other closed cosmologies.

Explicit calculations were used in the paper to show that ANEC is violated this way. Perturbative calculations show (in one of the explicit examples) that the wormhole remains traversable for longer and longer as the zero-temperature limit is approached, suggesting that a non-perturbative treatment would give an eternally traversable wormhole [45]. We direct the reader to reference [45] for more details.

5.2.5 Humanly traversable wormholes

The most recent paper on this topic, presenting perhaps the most surprising result until now, was published in August 2020 by Maldacena and Milekhin [46]. They managed to find a wormhole construction which is, in theory, not only traversable for particles and photons, but also for human beings. The base of this assembly is a previously considered theory for physics beyond the Standard Model: the Randall Sundrum II model. This is a model which describes the universe as a 5-dimensional anti-de Sitter (AdS_5) space where the elementary particles (except the graviton) are localized on a $(3+1)$-dimensional brane [47]. This can also be viewed as a 4-dimensional CFT coupled to gravity [46]. The same way 4-dimensional fermions were equivalent to 2-dimensional fermions under the action of the magnetic field of the near extremal black holes in the paper discussed in section 5.1, in this paper a 2-dimensional CFT will emerge from the 4-dimensional CFT in the presence of a magnetic field. The ANEC violation occurs due to a Casimir-like negative energy density. However, what looks like a quantum Casimir energy in four dimensions is actually a classical effect in five dimensions.

As opposed to the main paper discussed in this dissertation, some additional constraints need to be fulfilled in this case to allow the wormhole to be humanly traversable. The tidal acceleration felt by the infalling observer have to be smaller than the maximum sustainable accelerations for the human body, which are around $20g$ for brief periods of time. This gives a lower limit for the radius of the throat r_e and with it we can calculate an estimate of the lower limit of length of the wormhole l:

$$r_e > 1.5 \times 10^7 m$$

$$l > 3 \times 10^3 ly$$

It is noticeable that the wormhole's circumference needs to be very large to not crush the observer, but this case is not excluded by the theoretical model considered.

Section 6

Conclusion

This dissertation is a review of the most important concepts in wormhole physics and of the most recent recipes to construct these extreme objects. Briefly stated, traversable wormhole creation is dependent on violations of the averaged null energy condition (ANEC). The negative energy densities obtained as a result of these violations are the key to traversable wormholes. In the examples discussed above, ANEC violation was obtained either through a Casimir-like negative energy produced by fermions moving along magnetic field lines (sections 5.1 and 5.2.5), through an interaction between CFTs living on the boundaries of an eternal BTZ black hole (sections 5.2.1 and 5.2.2) or through backreaction of quantum fields (sections 5.2.3 and 5.2.4). If we desire a wormhole which is safe for human travel, additional restrictions related to the maximum tidal forces sustainable by the human body must be taken into consideration.

The duration for which the wormhole is traversable depends on the mechanisms employed to create it. There could exist eternally traversable wormholes (as discussed in section 5.2.2) or long-lived wormholes which are only traversable for a brief period of their existence (like the example provided in section 5.2.1). More than that, even the lifetime of a wormhole can be limited if the system is not stable (for example, if it loses energy through electromagnetic radiation or gravitational waves, or if small perturbations destabilize the entire structure and lead to collapse).

It is fascinating to observe how our knowledge of wormholes changed through time. It started with a simple solution to Einstein's field equations and it progressed to complex energy and topology conditions, quantum effects, questions about causality and time travel and detailed geometrical descriptions of plausible constructions. At first it took decades for any significant progress to be made, but nowadays a new revolutionary construction gets published every few months. While there is so much information available about wormholes, there is still so much we do not know about them.

First of all, even with the many existing descriptions of constructions of traversable wormholes, we are nowhere close to actually creating one, or at least to imagining an experiment which would, once put into practice, bring into existence such an object. Moreover, no wormholes have been observed so far in the Universe. All the detailed descriptions we discussed in this paper are purely theoretical models. We can only hope that our technology will advance fast enough for such experiments or observations to be possible in the future (after all, LIGO was something inconceivable 60 years ago).

Secondly, while quite a few ways of violating the averaged energy condition were mentioned in this paper, there might exist other, more elegant methods which are yet to be discovered and which would greatly simplify the creation of a traversable wormhole. There might also exist means other than the ones discussed in the examples above to stabilize the resulting wormhole, which could lead to longer-lived traversable wormholes.

Thirdly, the extent of the implications of traversable wormholes on quantum information theories and information transport is still unknown. We are familiar with the well known example of spooky action at a distance (entangled qubits), but adding patches of spacetime connected by wormholes, or coupled black hole boundaries (like the example in section 5.2.1) brings about new situations to be analyzed.

Lastly, since traversable wormholes are thought to appear as a result of both general relativity and quantum effects, advances in this domain could also represent advances towards a unified theory of quantum gravity. Thus, even if these constructions will not end up being useful from the practical perspective of long-distance travel, they might be the solution to one of the biggest unsolved problems in modern physics.

Construction of traversable wormholes is a field of physics in which, despite the recent progress and development occurring constantly, there are still many questions left unanswered. It is an exciting time for physicists working in this domain and one can only hope that the coming decades will bring even more wonderful revelations.

Bibliography

[1] Matt Visser. *Lorentzian Wormholes – From Einstein to Hawking*. American Institute of Physics Press, 1995.

[2] David Daring. Wormhole.

[3] David Hochberg and Matt Visser. Dynamic wormholes, antitrapped surfaces, and energy conditions. *Physical Review D*, 58:021–044, 1998.

[4] Amalkumar Raychaudhuri. Relativistic Cosmology. I. *Physical Review*, 98(4):1123–1126, May 1955.

[5] Ludwig Flamm. Comments on einstein's theory of gravity. *Physikalische Zeitschrift*, 17:448, 1916.

[6] Albert Einstein and Nathan Rosen. The particle problem in the general theory of relativity. *Physical Review*, 48:73, 1935.

[7] Fay Dowker. Lecture notes for the black holes course, qfff 2015.

[8] David Finkelstein. Past-future asymmetry of the gravitational field of a point particle. *Phys. Rev.*, 110:965–967, 1958.

[9] P. K. Townsend. Black holes, 1997.

[10] John Wheeler. Geons. *Physical Review*, 97:511–536, 1955.

[11] Homer G. Ellis. Ether flow through a drainhole: A particle model in general relativity. *Journal of Mathematical Physics*, 14:104–118, 1973.

[12] Kirill A. Bronnikov. Scalar-tensor theory and scalar charge. *Acta Physica Polonica*, B4:251–266, 1973.

[13] M. S Morris, K. S. Thorne, and U. Yurtsever. Wormholes, time machines, and the weak energy condition. *Physical Review Letters*, 61:1446–1449, 1988.

[14] M. S. Morris and K. S. Thorne. Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity. *American Journal of Physics*, 56:395–412, 1988.

[15] L. Landau and E. M. Lifshitz. *Classical theory of fields*. Pergamon Press, Oxford, UK, 1975.

[16] S. Hawking and G. Ellis. *The large scale structure of space-time*. Cambridge University Press, 1975.

[17] Roger Penrose. *Techniques of Differential Topology in Relativity*. Society for Industrial and Applied Mathematics, 1987.

[18] John L. Friedman, Kristin Schleich, and Donald M. Witt. Topological censorship. *Physical Review Letters*, 75(9):1872–1872, 1995.

[19] G. J. Galloway, K. Schleich, D. M. Witt, and E. Woolgar. Topological censorship and higher genus black holes. *Physical Review D*, 60(10), 1999.

[20] Gunnar Klinkhammer. Averaged energy conditions for free scalar fields in flat spacetime. *Physical Review D*, 43:2542–2548, 1991.

[21] A. Folacci. Averaged-null-energy condition for electromagnetism in Minkowski spacetime. *Physical Review D*, 46:2726–2729, Sep 1992.

[22] Rainer Verch. The averaged null energy condition for general quantum field theories in two dimensions. *Journal of Mathematical Physics*, 41(1):206–217, 2000.

[23] Thomas Faulkner, Robert G. Leigh, Onkar Parrikar, and Huajia Wang. Modular Hamiltonians for deformed half-spaces and the averaged null energy condition. *Journal of High Energy Physics*, (9), 2016.

[24] Thomas Hartman, Sandipan Kundu, and Amirhossein Tajdini. Averaged null energy condition from causality. *Journal of High Energy Physics*, (7), 2017.

[25] Jacob D. Bekenstein. Black holes and entropy. *Physical Review D*, 7:2333–2346, 1973.

[26] Aron C Wall. Ten proofs of the generalized second law. *Journal of High Energy Physics*, (06):021–021, 2009.

[27] Aron C. Wall. Proving the achronal averaged null energy condition from the generalized second law. *Physical Review D*, 81(2), 2010.

[28] Carlos Barceló and Matt Visser. Brane surgery: energy conditions, traversable wormholes, and voids. *Nuclear Physics B*, 584(1-2):415–435, 2000.

[29] Douglas Urban and Ken D. Olum. Averaged null energy condition violation in a conformally flat spacetime. *Physical Review D*, 81(2), 2010.

[30] Matt Visser. Gravitational vacuum polarization. II. Energy conditions in the Boulware vacuum. *Physical Review D*, 54(8):5116–5122, 1996.

[31] Matt Visser. Gravitational vacuum polarization. I. Energy conditions in the Hartle-Hawking vacuum. *Physical Review D*, 54:5103–5115, 1996.

[32] Ping Gao, Daniel Louis Jafferis, and Aron C. Wall. Traversable wormholes via a double trace deformation. *Journal of High Energy Physics*, (12), 2017.

[33] L. H. Ford and Thomas A. Roman. Averaged energy conditions and evaporating black holes. *Physical Review D*, 53(4):1988–2000, 1996.

[34] W. J. van Stockum. The gravitational field of a distribution of particles rotating about an axis of symmetry. *Proceedings of the Royal Society of Edinburgh*, 57:135–154, 1938.

[35] Kurt Gödel. An example of a new type of cosmological solutions of Einstein's field equations of gravitation. *Rev. Mod. Phys.*, 21:447–450, 1949.

[36] Frank J. Tipler. Causality violation in asymptotically flat space-times. *Phys. Rev. Lett.*, 37:879–882, 1976.

[37] S. W. Hawking. Chronology protection conjecture. *Physical Review D*, 46:603–611, 1992.

[38] Noah Graham and Ken D. Olum. Achronal averaged null energy condition. *Physical Review D*, 76(6), 2007.

[39] Juan Maldacena, Alexey Milekhin, and Fedor Popov. Traversable wormholes in four dimensions, 2018.

[40] L. Landau and E. M. Lifshitz. *Quantum Mechanics Non-Relativistic Theory. Third edition*. Butterworth-Heinemann, 1981.

[41] Máximo Bañados, Claudio Teitelboim, and Jorge Zanelli. Black hole in three-dimensional spacetime. *Physical Review Letters*, 69(13):1849–1851, 1992.

[42] Juan Maldacena and Xiao-Liang Qi. Eternal traversable wormhole, 2018.

[43] Gary T Horowitz, Don Marolf, Jorge E Santos, and Diandian Wang. Creating a traversable wormhole. *Classical and Quantum Gravity*, 36(20), 2019.

[44] Sidney Coleman. *Aspects of Symmetry: Selected Erice Lectures*. Cambridge University Press, Cambridge, U.K., 1985.

[45] Zicao Fu, Brianna Grado-White, and Donald Marolf. A perturbative perspective on self-supporting wormholes. *Classical and Quantum Gravity*, 36(4), 2019.

[46] Juan Maldacena and Alexey Milekhin. Humanly traversable wormholes, 2020.

[47] Lisa Randall and Raman Sundrum. An alternative to compactification. *Phys. Rev. Lett.*, 83:4690–4693, 1999.